Entropy, Gaussian Distribution and Fractional Processes

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Introduction

The concept of entropy as a measure of the chaos of a dynamical system is used in numerous applications:

- information theory,
- machine learning,
- information security,
- cryptography,
- environmental sciences,
- genetics,
- market analysis,
- climate analysis,
- social network analysis,
- chemical reactions,
- medical measurements.

Entropy plays a pivotal role in **signal processing and network traffic analysis.** It is employed in the development of algorithms for detecting **DDoS attacks, data compression, decision tree construction.** Fractional processes serve as essential models for capturing long-range dependence and self-similarity in diverse data types. Entropy plays a crucial role in quantifying the complexity and information content of signals generated by fractional processes, which proves invaluable for tasks like **prediction, risk assessment and anomaly detection.**

Methodology

- Investigation of the properties of the various entropies for the centered Gaussian distribution with respect to the parameters was generally done by applying **calculus methods.**

- It is worth noting that certain theoretical properties, particularly the convexity of the Tsallis entropy, are challenging to analyze analytically. In such cases, we employ **numerical investigations**, which offer insights into theoretical properties.

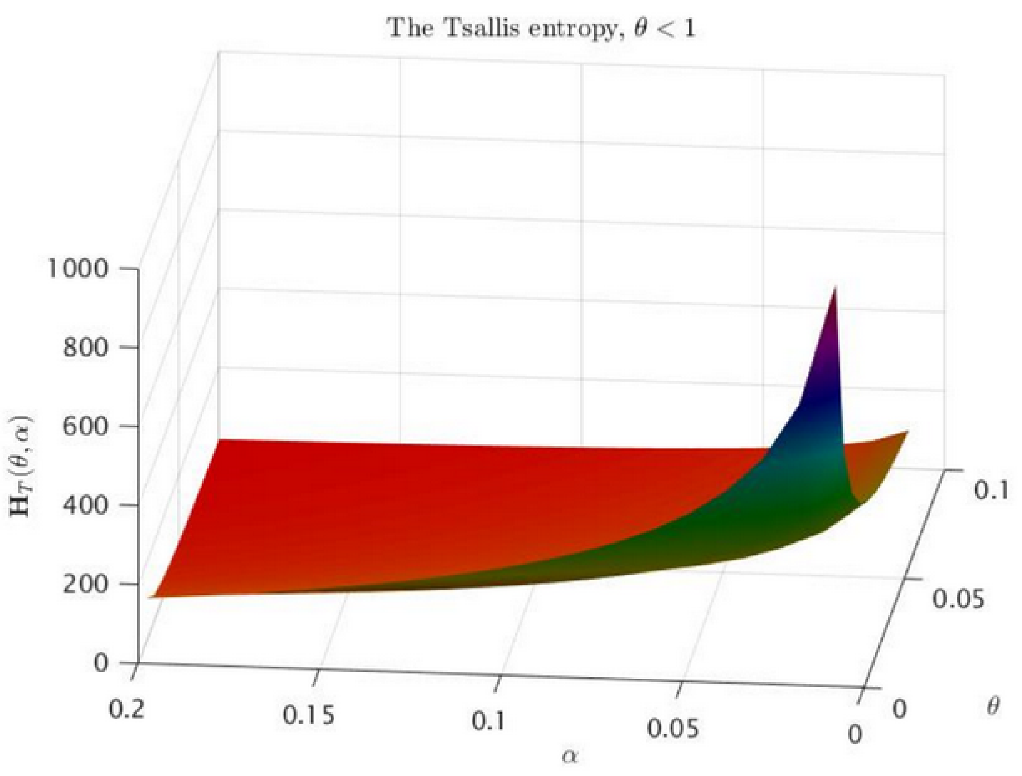

Figure 1. Tsallis entropy as a function of θ and α, $\theta < 1$.

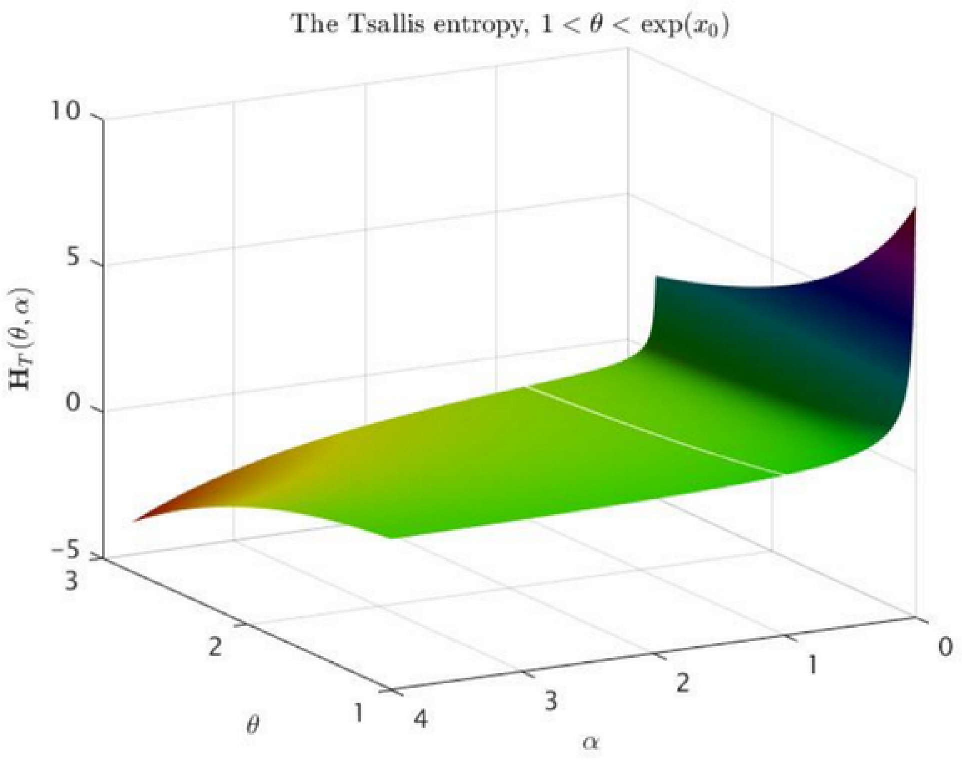

Figure 3. Tsallis entropy as a function of θ and α, $1 < \theta < e^{x_0}$.

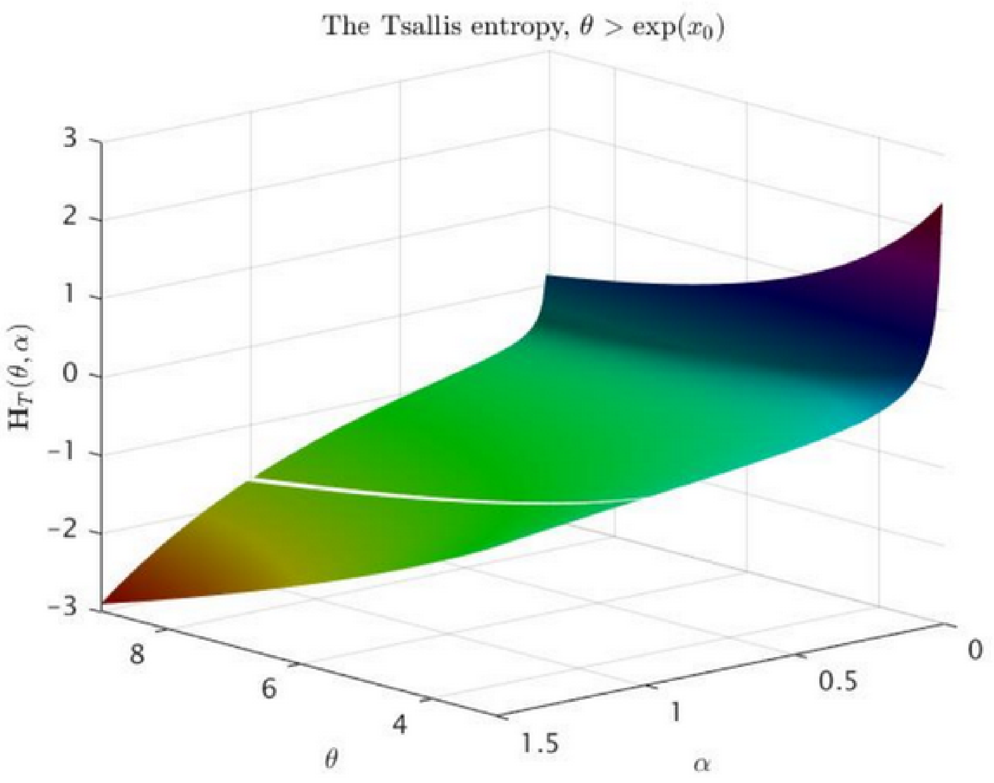

Figure 4. Tsallis entropy as a function of θ and α, $\theta > e^{x_0}$.

Results

We compare the entropies of the one-dimensional distributions of the following fractional processes: fractional Brownian motion, subfractional Brownian motion, Riemann–Liouville fractional Brownian motion, bifractional Brownian motion and three types of multifractional Brownian motion (moving-average, Volterra-type and harmonizable), as well as tempered fractional Brownian motions of the first and second kind.

We consider **normalized versions** of these processes to ensure their variances at t=1 to be equal 1. After this normalization, we observe that fractional Brownian motion, subfractional Brownian motion and Riemann–Liouville fractional Brownian motion **share the same entropies**. Similar formulas apply to bifractional Brownian motion; furthermore, its entropies can be compared to those of fractional Brownian motion depending on the values of t.

For multifractional Brownian motion, we have established that the **moving-average and harmonizable versions of this process have the same entropies**. These entropies can be compared with the corresponding entropies of Volterra-type multifractional Brownian motion, depending on the behavior of the Hurst function.

Lastly, for two versions of tempered fractional Brownian motions, we can **numerically** compare their entropies depending on the ratio between the multiplicative constants involved in their definitions.

Future research

Our research opens up possibilities for future extensions in several directions.

Potential avenues for further investigation include exploring various entropy measures for **non-Gaussian processes, nonstationary processes and processes with nonstationary increments.** Additionally, we can delve into the solutions of stochastic differential equations that describe the interactions of particle systems within random environments.

Conclusion

- We examined five distinct entropy measures applied to the Gaussian distribution: Shannon entropy, Rényi entropy, generalized Rényi entropy, Tsallis entropy and Sharma−Mittal entropy. We investigated **their interrelationships and analyzed their properties in terms of their dependence on specific parameters.**
- Our study extends to fractional Gaussian processes, encompassing fractional Brownian motion, subfractional Brownian motion, bifractional Brownian motion, multifractional Brownian motion and tempered fractional Brownian motion. We conducted a **comparative analysis** of the entropies associated with the one-dimensional distributions of these processes.

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